International Journal of Heat and Mass Transfer 62 (2013) 230–241

Contents lists available at SciVerse ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Analysis of the estimation error in a parsimonious temperature-temperature characterization technique

Denis Maillet *, Yves Jannot, Alain Degiovanni

Université de Lorraine, LEMTA, UMR 7563, 2, Avenue de la Forêt de Haye, TSA 60604 - 54518 Vandoeuvre-lès-Nancy Cedex, France
CNRS, LEMTA, UMR 7563, 2, Avenue de la Forêt de Haye, TSA 60604 - 54518 Vandoeuvre-lès-Nancy Cedex, France

ARTICLE INFO

Article history:
Received 24 August 2012
Received in revised form 9 January 2013
Accepted 19 February 2013

Keywords:
Thermal characterization
Thermal impedance
Convolution product
Transfer function
Parameter estimation
Standard deviation and bias
Parsimony

ABSTRACT

Characterization of systems often relies on a single point response that is a convolution product between a transfer function, an impedance, and the strength of a source. This requires perfect knowledge or measurement of this source. Another possibility is to measure two point responses in such systems. These are also linked, under certain causality conditions, by a different convolution product based on another type of transfer function, called here a transmittance. It is shown that using this kind of transmittance-based model, with one of the two responses as a pseudo-source to explain the other one, leads to a model with fewer parameters, which is very interesting for parameter estimation. Replacement of the exact strength of the source by a noised response in a non-linear least square minimization process does not bring any additional bias and the standard deviations of the parameter estimates can be calculated on a theoretical stochastic basis. An example of such an estimation technique in a thermal characterization of a light insulating sample by the three-layer method is used to show the practical interest of this estimation method and to validate the assessment of the estimation errors through a Monte Carlo approach. Finally a counting of the number of parameters present in the transmittance model, within a very general context of a one-dimension heat transfer characterization experiment, shows its parsimonious character when compared with impedance-based parameter estimation techniques.

1. Introduction

Classical transient thermal characterization techniques rely on a common principle: a sample, at uniform initial temperature, is stimulated by a surface heat source $q(t)$ (W m^{-2}) which is applied onto one of its faces, while its transient temperature rise $T(t)$ (kelvins), at a single given point, is recorded. Let us note that variations exist where $q(t)$ is a lineic heat source (W m^{-1}), for the hot wire method) or even a power heat source (W, for the spherical hot probe method).

The value of the unknown parameters, called β_j (for $j = 1$ to n) here, are estimated next through a minimization, in the least square sense, of the difference between the experimental temperature response $T^{\exp}(t)$ and the theoretical output of the corresponding modelled response $T^{\text{mod}}(t;\boldsymbol{\beta})$, where parameter vector $\boldsymbol{\beta}$ gathers the β_j parameters.

Examples of such characterization techniques, here the flash method [1] and the hot plane method [2,3], whose principles are recalled in Figs. 1 and 2, will be detailed in the next two sections.

In this class of methods based on models of the q to T type, which we will call A models now on, only one temperature $T^{\exp}(t)$ is measured while some information about the source term has to be known. This knowledge can have two different origins:

- In methods of the A1 type, the variation of $q(t)$ with time is completely known. This heat source is a Heaviside function (step) for the transient hot plane method. The effect of temperature measurement noise errors on the estimated parameters is now completely known for A1 methods because the estimated variance–covariance matrix depends on the standard deviation of the noise and on the sensitivity matrix of the temperature model with respect to the parameters which are looked for, for an independent identically distributed noise, see [4,5], and [6].
- In methods of the A2 type, this shape is known, within a multiplicative constant that has to be estimated, together with the other β_j parameters: in the case of the flash method, see Section 1.1.
- In methods of the A3 type, the intensity $q(t)$ source is completely unknown, so this signal has to be measured in addition to the temperature signal $T^{\exp}(t)$.

* Corresponding author at: Université de Lorraine, LEMTA, UMR 7563, 2, Avenue de la Forêt de Haye, TSA 60604 - 54518 Vandoeuvre-lès-Nancy Cedex, France. Tel.: +33 03 83 59 56 06/07; fax: +33 03 83 59 55 51.

E-mail addresses: denis.maillet@univ-lorraine.fr (D. Maillet), yves.jannot@univ-lorraine.fr (Y. Jannot), alain.degiovanni@univ-lorraine.fr (A. Degiovanni).

0017-9310/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijheatmasstransfer.2013.02.055

Nomenclature

a	thermal diffusivity ($m^2\,s^{-1}$)
$\mathbf{b}_{\hat{\beta}}$	estimation bias vector
c	specific heat ($J\,kg^{-1}\,K^{-1}$)
C_1,C_2	thermal capacities, for a unit area, of brass layers ($J\,m^{-2}\,K^{-1}$)
cov (·)	variance–covariance matrix of a random vector
d	distance between sensors in the thin plate method (m)
e	thickness of sample (m)
$\mathbf{e}_{\hat{\beta}}$	estimation error vector
E	thermal effusivity ($W\,s^{1/2}\,m^{-2}\,K^{-1}$)
$E(\cdot)$	mathematical expectancy
$f(\cdot)$	function
$g(t)$	time function corresponding to the shape of the intensity of the source
h	heat transfer coefficient ($W\,m^{-2}\,K^{-1}$)
$H(\cdot)$	Heaviside function
$\mathbf{I}_r$	identity matrix of order r
J_w	least squares sum based on a transmittance model
J_z	least squares sum based on an impedance model
$\mathbf{M}(\cdot)$	convolution matrix function for a column vector
N	number of synthetic experiments for Monte Carlo simulations of estimations
m	number of times of measurement
n	number of parameters
n_{Ap}	number of parameters to be estimated in estimations of the Ap type (p = 1 to 3)
n_B	number of parameters to be estimated in estimations of the B type
n_z	number of parameters in the temperature field solution
$n_{z\ i}$	number of parameters in parameter vector $\boldsymbol{\beta}_{z\ i}$
p	Laplace parameter (s^{-1})
$q(\cdot)$	intensity of a surface source depending on time (W or $W\,m^{-2}$)
q_0	coefficient of a surface source depending on time (W or $W\,m^{-2}$)
Q_0	energy of a surface source ($J\,m^{-2}$)
$\mathbf{r}$	residual vector (K)
R_c	thermal contact resistance ($K\,m^2\,W^{-1}$ or $K\,W^{-1}$)
$\mathbf{S}_w$	sensitivity matrix of transmittance vector $\mathbf{W}$ with respect to parameter vector $\boldsymbol{\beta}$ (dimensions: $m \times n$)
s	statistical standard deviation of a parameter estimate
t	time (s)
t_0	duration of a door excitation (s)
t_k	kth time of measurement (s)
T	temperature (K)
T_1	temperature of reference sensor (K)
T_2	temperature of response (K)
$\mathbf{T}$	temperature vector (K)
$\mathbf{x}$	position vector in 1D, 2D or 3D (m)
$W(\cdot)$	transmittance transfer function (–)
$\mathbf{W}$	transmittance vector ($K\,W^{-1}$ or $K\,m^{-2}\,W^{-1}$)
$\mathbf{X}$	sensitivity matrix of temperature vector $\mathbf{T}$ with respect to parameter vector $\boldsymbol{\beta}$ (dimensions: $m \times n$)
$Z(\cdot)$	impedance transfer function ($K\,W^{-1}$ or $K\,m^{-2}\,W^{-1}$)
$\mathbf{Z}$	impedance vector (-)

Symbols

β_j	parameter number j
$\boldsymbol{\beta}$	parameter vector in the transmittance transfer function
$\boldsymbol{\beta}_z$	parameter vector in the temperature field solution
$\boldsymbol{\beta}_{z\ i}$	parameter vector in the impedance transfer function number i (i = 1 or 2)
$\delta(t)$	Dirac distribution in time (s^{-1})
$\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$	noise column vectors for temperature sensors 1 and 2 (K)
$\boldsymbol{\varepsilon}^*$	concanated noise column vector (K)
ρ	density ($kg\,m^{-3}$)
σ	stochastic standard deviation of measurement noise (K)
τ	characteristic diffusion time (s)
$*$	convolution product between two functions of time

Subscripts

i	number of a sensor
j	number of a scalar parameter in a parameter vector
k	number of a measurement time
phys	physical

Superscripts

avail	available value for assessment of stochastic properties of estimated parameters
exact	exact value of a parameter
exp	experimental
true	true (unbiased) model
T	transpose of a matrix
^	estimated value or estimator of a parameter
–	Laplace transform
$*$	dimensionless

In order to get rid of the constraints associated to characterization techniques of the A2 or A3 type, another very attractive approach consists in measuring the temperatures (1 and 2) at two different locations instead of one. These models can be designated by T_1 to T_2 models or called also B models here. Once the two temperatures measured, the corresponding signals have to be processed through a least square minimization.

Different examples can be given for class B methods: the thin plate method [7,8] will be briefly discussed in Section 1.3 while the three-layer method [9] will be presented in Section 1.4. Another example of this class of method, in cylindrical geometry, is the parallel hot wire technique [10,11].

So, thermal characterization methods of the class B type now exist but, to our knowledge, no stochastic study has ever been implemented to derive the standard deviations of the estimation error in the same way as what has been done for class A1 methods. That is the objective of what will be developed in Section 2.

1.1. The flash method

In the flash method case (A2 type method, see [1] for example), the "rear face" temperature response T_2 of a homogeneous sample

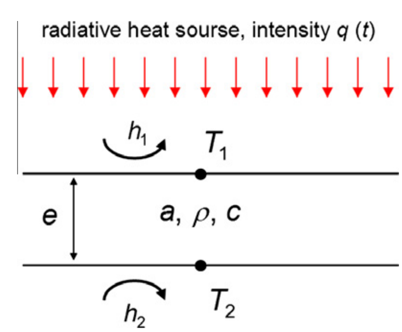


Fig. 1. Principle of the flash method.

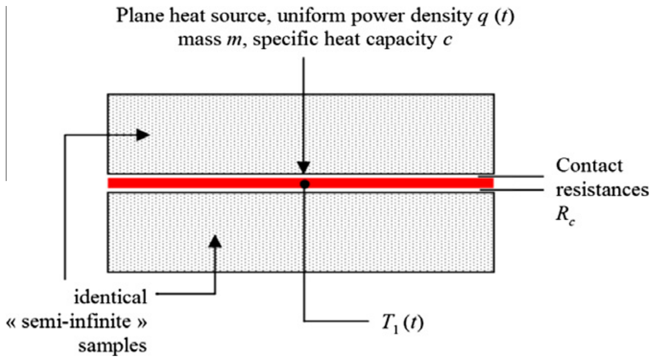


Fig. 2. Principle of the hot plane method.

of thickness, to a front face surface heat source, is measured and its thermal diffusivity a is looked for using a transient conduction model, see Fig. 1. The sample thermo physical properties are supposed not to depend on temperature during the experiment.

The following additional assumptions can be made here:

- the surface heat source is a pulse one: $q(t) = Q_0\delta(t)$, with $\delta(t)$ the Dirac distribution (s^{-1}) and Q_0 (J/m^2) the excitation level, that is the energy absorbed by one unit of area of the sample;
- heat transfer is one-dimensional with two equal convection constant coefficients ($h_1 = h_2 = h$) and the surrounding ambient air temperature is equal to the uniform initial temperature of the sample (taken equal to zero here);

With these conditions the heat equation and its boundary conditions, are linear and the temperature response becomes:

$$T_2 = f_{phys}(t; a, h, \rho, c, e, q(t)) \quad (1)$$

where f_{phys} is a function of time t, on the geometrical (e) and thermophysical properties (diffusivity a and volumetric heat ρc) of the sample and on its coupling with the outside environment at a zero reference temperature level through the h coefficient. This function also depends on the intensity $q(t)$ of the source in a linear way. It can be called a "physical" function here since its arguments, its variable and its value have physical units.

A close look at the way the parameters are present, either in the heat equation and its associated conditions or in its solution (5), allows to identify the parameter groups whose knowledge is compulsory for the calculation of the output of the T_2 temperature model:

$$T_2 = f(t; a/e^2, he/\lambda, Q_0/\rho ce) \quad (2a)$$

A normalization of both the argument (time) and of its value (temperature) of this new function f, to make the Fourier number t^* and the reduced temperature T_2^* (with $0 \leqslant T_2^* \leqslant 1$) appear, transforms the previous equation into:

$$T_2^* = f^*(t^*; he/\lambda) \quad \text{with} \quad t^* = (a/e^2)t \quad \text{and} \quad T_2^* = \frac{T_2}{Q_0/\rho ce} \quad (2b)$$

Let us note that only function f^* in Eq. (2b) is a real mathematical function, that is of dimensionless value and linking dimensionless quantities. Consequently the three parameters present as arguments of function f in Eq. (2a) are independent groups which appear at different locations in the heat equation and in its associated boundary conditions or in the analytical expression of its solution. They are

- either "pi groups" (see Vaschy-Buckingham theorem) in the dimensional analysis of this problem (case of the Biot number $h\,e/\lambda$),
- or normalization constant (s) (case of the characteristic frequency a/e^2 and of the adiabatic asymptotic temperature $Q_0/\rho c\,e$) necessary to obtain the mathematical model (2b).

This shows that estimation of ρc, the sample volumetric capacity, as an additional parameter, would require the measurement of Q_0, the energy absorbed by unit area of the front face, which corresponds to a characterization method of the A3 type.

So the temperature solution at any point in the sample is a convolution product (noted $*$ here) between the heat source $q(t)$ and a transfer function Z_2:

$$T_2(t; \boldsymbol{\beta}) = Q_0 Z_2(t; \beta_1, \beta_2, \rho ce) = \beta_3 Z_2^*(t; \beta_1, \beta_2)$$
$$\text{where } \boldsymbol{\beta} = \left[\frac{a}{e^2}, \frac{h_2 e}{\lambda}, \frac{Q_0}{\rho ce}\right]^T \quad \text{and} \quad Z_2^* = \rho ce Z_2 \quad (3)$$

where superscript T designates the transpose of a matrix here.

Another very interesting form exists for Eq. (3), if the Laplace transforms of the three preceding quantities are considered:

$$\overline{T}_2(p; \boldsymbol{\beta}) = \overline{Z}_2(p; \beta_1, \beta_2, \rho ce)\bar{q}(p) \quad (4)$$

where

$$\bar{x}(p) = \int_0^\infty x(t)\exp(-pt)\mathrm{d}t \quad \text{for } x = T_2, Z_2 \text{ or } q$$

Transfer function Z_2 relies a temperature T_2, that is a potential or a potential difference, to the heat source q (here a power surface density, but it can also be a thermal power) is analogous to a current density in electricity (or the intensity of the corresponding current). It can be called thermal "impedance".

1.2. The hot plane method

In the hot plane method (A1 type method, see [2]) case the "front face" temperature (heated face) T_1 is measured, which respects the following model:

$$T_1(t; \boldsymbol{\beta}) \equiv Z_1(t; \boldsymbol{\beta}) * q(t) \quad \text{with } \boldsymbol{\beta} = [E \quad R_c \quad mc]^T \quad (5)$$

Here 3 parameters, thermal effusivity E, contact resistance R_c and heat capacity mc of the heating probe (heat source) are looked for, with the following assumptions:

- the surface heat source $q(t)$ (W/m^2) is perfectly known;
- the thicknesses of both samples are considered to be infinite (semi-infinite heat transfer).

1.3. The thin plate method

The thin plate method [7,8], is designed for measuring the thermal diffusivity a of thin samples, in a strip shape, see Fig. 3, in the direction of their plane. A transient heat source $q(t)$ of arbitrary time shape is set over part of the sample surface and temperatures T_1 and T_2 at two different locations of the sample are measured. So, it is a method of the B type. It relies on the fin assumption for the plate:

- the plate sample is homogeneous and isotropic in the direction of its plane, with λ its corresponding thermal conductivity;
- the Biot number $Bi = he/\lambda$ associated to the heat losses caused by convection to the surrounding air and linearized radiation to the radiative environment, at the same temperature T_∞ as air, is much smaller than unity. h represents the heat transfer coefficient of these losses on both faces of the fin. It is supposed to be uniform and e is the plate thickness;
- the initial temperature of the sample, before the start of $q(t)$ is equal to the reference temperature of the heat losses T_∞;

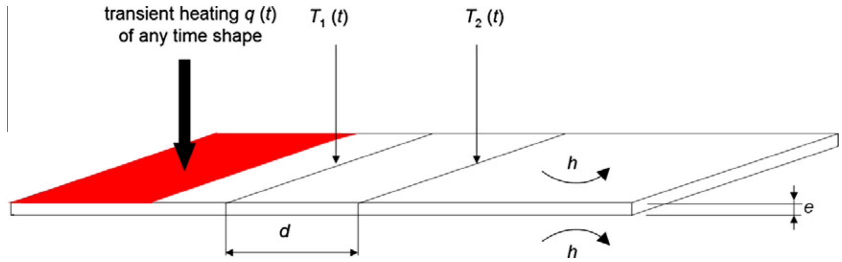


Fig. 3. Principle of the thin plate method.

– the temperature distribution is one-dimensional and the plate length is large enough to consider the fin as semi-infinite.

The corresponding model writes out:

$$T_2(t;\boldsymbol{\beta}) \equiv W_{21}(t;\boldsymbol{\beta}) * T_1(t) \quad \text{with } \boldsymbol{\beta} = [\tau \quad b]^T$$
$$\text{where } \tau = d^2/a \text{ and } b = Bid^2/e^2 \tag{6}$$

Here the new transfer function W_{21} can be called a "transmittance". Its Laplace transform takes a very simple form:

$$\overline{W}_{21}(p;\boldsymbol{\beta}) = \exp\left(-\sqrt{\tau p + 2b}\right) \tag{7}$$

The interest of this model is that measurement of $q(t)$ is not required and it involves two parameters only: the characteristic time τ associated to the distance between the two temperature sensors, which allows the estimation of the in-plane diffusivity a, and parameter b associated with both the Biot number and the geometric shape factor d/e.

1.4. The three-layer method

An other method of the B type, the three-layer method [9], see Fig. 4, has been proposed recently: it relies on the transient measurement of temperatures at two different locations of the sample and on the estimation of the unknown parameters present in the transfer function W_{21} between these two temperatures, called "transmittance" here. The corresponding model writes out:

$$T_2(t;\boldsymbol{\beta}) \equiv W_{21}(t;\boldsymbol{\beta}) * T_1(t) \quad \text{with } \boldsymbol{\beta} = [a \quad \lambda \quad h_2]^T \tag{8}$$

Here three parameters, thermal diffusivity a, thermal conductivity λ and heat exchange coefficient h_2 are looked for.

Comparison of Eq. (8) to the A type model (5) shows that, in this new configuration, the heating resistance generates a surface power source $q(t)$ which is the cause of both responses T_1 and T_2 of the brass layers through two impedances Z_1 and Z_2 such as (in the Laplace domain):

$$\overline{T}_1 = \overline{Z}_1\bar{q} \quad \text{and} \quad \overline{T}_2 = \overline{Z}_2\bar{q} \tag{9a, b}$$

Elimination of $\bar{q}$ between Eqs. (9a) and (9b) yields:

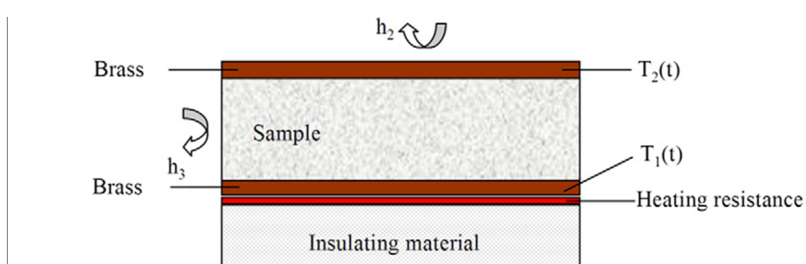
$$\overline{T}_2 = \overline{W}_{21}\overline{T}_1 \quad \text{with } \overline{W}_{21} = \overline{Z}_2/\overline{Z}_1 \tag{10}$$

Return to the time domain leads directly to model (6). Here it can be shown that the transfer function W_{21} in the time domain exists (the transmittance function), while the opposite ratio $\overline{Z}_1/\overline{Z}_2$ is not the Laplace transform of any function: W_{12} does not exist.

So, when comparing Eqs. (10) and (8), one can consider that the real cause $q(t)$ of temperature response T_2 has been replaced by a "pseudo" cause $T_1(t)$.

As a consequence, estimation of the three parameters present in model (8) can be estimated through minimization of the least square sum based on the difference between the experimental $T_2^{\exp}(t)$ response and the modelled one, $W_{21}(t;\boldsymbol{\beta}) * T_1(t)$, corresponding to (8) where the theoretical pseudo-excitation $T_1(t)$ has been replaced by its measured value $T_1^{\exp}(t)$.

This method of estimation based on a *transmittance* type of transfer function allows a decrease of the number of parameters to be estimated with respect to the classical type of estimation



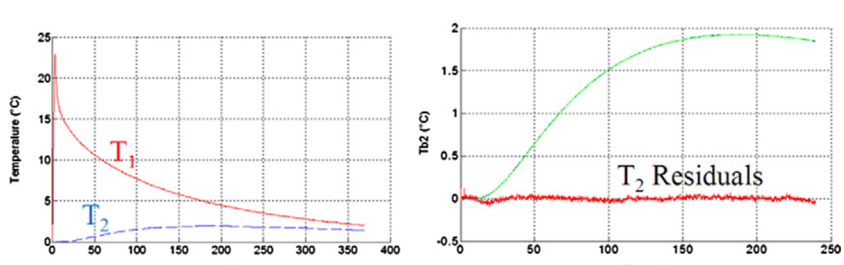


Fig. 4. Principle of the three-layer method, with the two temperature responses and parameter estimation based on T_2 response.

based on transfer functions of the *impedance* type: this increases the precision of the estimations.

In so doing, one gets rid of the need to know or to estimate the parameters that characterize phenomena occurring upstream the measurement point, that is the $T_1(t)$ measurement, either h_1 and $q(t)$ in the case of a surface source created by the absorption of a radiation (flash method) or a contact resistance R_c, a heat capacity mc and the intensity of the surface heat source $q(t)$ if this source is produced by a heating element in contact with the surface (hot plane method). No prior knowledge (assumptions about the time shape) or measurement of the source $q(t)$ is necessary anymore, which constitutes another advantage, compared to classical methods. One can give this source the most suited time shape in order to be able to get maximum sensitivities to the parameters that are to be estimated without the necessity to measure it.

In the next part of this paper we will not deal with the problems of the A3 type, where not only one temperature but also a transient heat source has to be measured concomitantly and we assume that:

- the models used for estimation are not biased,
- the values of parameters (or functions) that are not estimated, the 'parameters supposed to be known', are exact (including the intensity of the source in the A1 type of problem or its time shape for problems of the A2 type),
- the temperature measurements are made in a non intrusive way,
- the parameters of the temperature sensor model, that is the calibration curve that allows a conversion of the sensor output into a temperature, are known.

The parameters of the sensors are the emissivity of the sample surface in the case of a radiative measurement, or the thermoelectric power and the cold junction temperature in the case of a measurement by a thermocouple.

We will deal next with the problem of estimating the parameters present in the transmittance function $W_{21}(t;\boldsymbol{\beta})$ and we will focus on one part of the estimation error, that is the estimation error produced by the measurement noise only.

2. Least-square method for transmittance-based models and standard deviations of the estimates

2.1. Transmittance-based models and their parameter vectors

We consider here an impedance-based model of the A type with a single excitation $q(t)$ (also called a source, or an "input", in the signal processing sense). This can be a power, or a surface or linear power density, which is the common cause of the temperature responses in two different points of the physical system, through a convolution-type equation similar to Eq. (5):

$$T_1(t,\boldsymbol{\beta}_{z1}) \equiv Z_1(t,\boldsymbol{\beta}_{z1}) * q(t) \text{ and } T_2(t,\boldsymbol{\beta}_{z2}) = Z_2(t,\boldsymbol{\beta}_{z2}) * q(t) \tag{11a, b}$$

where $\boldsymbol{\beta}_{zi}$ is the parameter vector composed of n_{zi} parameters, and $Z_i(t;\boldsymbol{\beta}_{zi})$, for $i = 1$ or 2, the time *impedance*, that is the transfer function, between $q(t)$ and each temperature T_1 and T_2.

Both impedances depend on the structural parameters of the system and of the location of the two observations, which are the components of parameter vectors $\boldsymbol{\beta}_{zi}$.

The usual approach for estimating $\boldsymbol{\beta}_{z2}$ using the single T_2 measurement and *impedance*-based model (11b), for a known $q(t)$ source, is to minimize the ordinary least square sum [4]:

$$J_z(\boldsymbol{\beta}_{z2}) = \sum_{k=1}^{m} \left(T_2^{\exp}(t_k) - (Z_2(\boldsymbol{\beta}_{z2}) * q)(t_k)\right)^2 \tag{12}$$

where m is the number of available measurements for T_2.

In methods of the A1 type, T_2 is measured and $q(t)$ is perfectly known, so the estimation error $\hat{\boldsymbol{\beta}}_{z2} - \boldsymbol{\beta}_{z2}^{exact}$ is a random variable of zero mean and its variance–covariance matrix can be calculated, using the notions of sensitivity coefficients, scaled or not, gathered in a sensitivity matrix $\boldsymbol{S}$, even for *non linear* models (in the parameter estimation sense: Z_2 is a non linear function of $\boldsymbol{\beta}_{z2}$)) [4–6]:

$$\mathrm{cov}(\hat{\boldsymbol{\beta}}_{z2}) = \sigma^2(\boldsymbol{S}^T\boldsymbol{S})^{-1} \tag{13}$$

where σ is the standard deviation of the temperature measurement noise, which is assumed to be an independent identically distributed (i.i.d.) random variable here.

In the experiments of the B type both temperatures T_1 and T_2 are measured, impedance models (12a) and (12b) can be replaced by a single transmittance-based model, see Eqs. (6) and (10) above, where the output T_2 is the same but with a substitution of input $q(t)$ by T_1:

$$T_2 = T_2^{true}(t,\boldsymbol{\beta}) \equiv W_{21}(t,\boldsymbol{\beta}) * T_1(t) \tag{14}$$

Two important remarks have to be made here about this class of *transmittance*-based models (14):

(i) Temperature function T_1 has only a single argument, time t without any explicit presence of parameter vector $\boldsymbol{\beta}_{z2}$.
(ii) The second argument of transmittance W_{21}, and as a consequence of T_2, $\boldsymbol{\beta}$, is not $\boldsymbol{\beta}_{z2}$ anymore. Discussion about the respective sizes of both vectors, that is about the number of parameters presents in $\boldsymbol{\beta}_{z2}$ and $\boldsymbol{\beta}$ is made further on, in Section 4 of this article.

2.2. Matrix/vector form of a transmittance-based model and of its least square sum

In any application of a least square parameter estimation problem, the measurements $T_2^{\exp}(t_i)$ are made for discrete time values $t_k = k\Delta t$ over a $[t_0 = 0 \quad t_{end} = t_m = m\Delta t]$ time interval. So, both temperatures of model (14) can be vectorized on the corresponding discrete time grid:

$$\boldsymbol{T}_i = [T_{i_1} \quad T_{i_2} \quad \cdots \quad T_{im}]^T \text{ for } i = 1 \text{ or } 2 \text{ with}$$
$$T_{1_k} = T_1(t_k) \text{ and}$$
$$T_{2_k} = T_2(t_k;\boldsymbol{\beta}) \text{ for } k = 1 \text{ to } m \tag{15}$$

In exactly the same way, transmittance function $W_{21}(t,\boldsymbol{\beta})$, which will be noted W hereon, can be discretized into a column vector form:

$$\boldsymbol{W} = [W_1 \quad W_2 \quad \cdots \quad W_m]^T \text{ with } W_k = W(t_k;\boldsymbol{\beta}) \text{ for } k = 1 \text{ to } m \tag{16}$$

So, at the model level, convolution product (14) becomes the product of a square lower diagonal matrix by a column vector:

$$\boldsymbol{T}_2 = \mathbf{M}(\boldsymbol{W})\boldsymbol{T}_1 = \mathbf{M}(\boldsymbol{T}_1)\boldsymbol{W} \tag{17}$$

where $\mathbf{M}(\cdot)$ is a (square) matrix function of a column vector, here a Toeplitz matrix defined by:

$$\mathbf{M}(\boldsymbol{x}) \equiv \Delta t \begin{bmatrix} x_1 & & & & \\ x_2 & x_1 & & 0 & \\ x_3 & x_2 & x_1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ x_m & x_{m-1} & x_{m-2} & \cdots & x_1 \end{bmatrix} \quad \text{where } \boldsymbol{x} = \begin{bmatrix} x_1 = x(t_1) \\ x_2 = x(t_2) \\ x_3 = x(t_3) \\ \vdots \\ x_m = x(t_m) \end{bmatrix} \tag{18}$$

The least square sum based on the transmittance-based model can be written in a scalar form analogous to (11) or in its corresponding matrix/vector form:

$$J_w(\boldsymbol{\beta}) = \sum_{k=1}^{m}\left(T_2^{\exp}(t_k) - \left(W(\boldsymbol{\beta}) * T_1^{\exp}\right)(t_k)\right)^2 = \left\|\boldsymbol{T}_2^{\exp} - \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}))\boldsymbol{T}_1^{\exp}\right\|^2 \quad (19)$$

In this equation $\|.\|$ designates the L_2 norm of a column vector, with $\|\boldsymbol{z}\|^2 \equiv \boldsymbol{z}^T\boldsymbol{z} \equiv \sum_{k=1}^{m} z_k^2$ and $\boldsymbol{T}_1^{\exp}$ and $\boldsymbol{T}_2^{\exp}$ are the $(m \times 1)$ column vectors corresponding to discretized experimental measurements of both $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$, with:

$$\boldsymbol{T}_1^{\exp} = \boldsymbol{T}_1^{exact} + \boldsymbol{\varepsilon}_1 \text{ and } \boldsymbol{T}_2^{\exp} = \boldsymbol{T}_2^{true}(\boldsymbol{\beta}^{exact}) + \boldsymbol{\varepsilon}_2 \quad (20a)$$

$$\mathrm{E}(\boldsymbol{\varepsilon}_1) = \mathrm{E}(\boldsymbol{\varepsilon}_2) = 0 \text{ and } \mathrm{cov}(\boldsymbol{\varepsilon}_1) = \mathrm{cov}(\boldsymbol{\varepsilon}_2) = \sigma^2\boldsymbol{I}_m \quad (20b)$$

In the above equations, $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are the noise vectors, at the measurement times, on each temperature signal. They are supposed to be independent and identically distributed. E (.) and cov (.) are the mathematical expectancy and the variance–covariance matrix of random vector while $\boldsymbol{I}_m$ is the identity matrix of size m and σ the standard deviation of the measurement noise.

We also assume that both $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ noises are independent:

$$\mathrm{cov}(\boldsymbol{\varepsilon}^*) = \sigma^2\boldsymbol{I}_{2m} \quad \text{with } \boldsymbol{\varepsilon}^* = \begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix} \quad (20c)$$

Let us stress here that a constraint exists: the two measurement devices for T_1 and T_2 should be run in a synchronous mode (same acquisition period). We also assume, for facility reasons, that the two acquisitions are implemented with the same gain, in order to get the same standard deviation for both temperature signals: this yields an explicit expression of the level of the standard deviation of the estimates in Section 2.4 further down. However, more general cases, with different standard deviations, can also be considered.

Criterion (19) can be given an alternate form:

$$J_w(\boldsymbol{\beta}) = \left\|\boldsymbol{T}_2^{\exp} - \boldsymbol{T}_2^{avail}(\boldsymbol{\beta})\right\|^2 \quad \text{where } \boldsymbol{T}_2^{avail}(\boldsymbol{\beta}) \equiv \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}))\boldsymbol{T}_1^{\exp} = \mathbf{M}(\boldsymbol{T}_1^{\exp})\boldsymbol{W}(\boldsymbol{\beta}) \quad (21a, b)$$

Let us note that only an approximated version, $\boldsymbol{T}_2^{\mathrm{avail}}(\boldsymbol{\beta})$ of the output $\boldsymbol{T}_2^{true}(\boldsymbol{\beta})$ given by Eq. (14) is available here, with a noised input $\boldsymbol{T}_1^{\exp}$ instead of $\boldsymbol{T}_1^{exact}$.

It can be shown, see [6, Section 3.3.3.4]), that iterative minimization of $J_w(\boldsymbol{\beta})$, once put under a quadratic form around any nominal value of $\boldsymbol{\beta}$, allows getting an estimate which corresponds to a first order approximation:

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{exact} \approx \mathbf{A}(\boldsymbol{\beta}^{exact})\left(\boldsymbol{T}_2^{\exp} - \boldsymbol{T}_2^{avail}(\boldsymbol{\beta}^{exact})\right) \quad \text{where } \mathbf{A}(\boldsymbol{\beta}) = (\boldsymbol{X}^T(\boldsymbol{\beta})\mathbf{X}(\boldsymbol{\beta}))^{-1}\mathbf{X}^T(\boldsymbol{\beta}) \quad (22a, b)$$

Here the nominal value used for this series expansion of $J_w(\boldsymbol{\beta})$ is the exact value $\boldsymbol{\beta}^{exact}$, with the implicit assumption of a convergence of the minimization algorithm towards this value. $\boldsymbol{X}(\boldsymbol{\beta})$ is the matrix of the sensitivity coefficients of model (21b) with respect to the parameters present in $\boldsymbol{\beta}$.

2.3. Calculation of the different sensitivity matrices

The sensitivity matrix $\boldsymbol{X}^{true}(\boldsymbol{\beta})$ of $\boldsymbol{T}_2^{true}$, that is the *transmittance*-based model (14), with respect to the parameters present in $\boldsymbol{\beta}$ is introduced [5]:

$$\boldsymbol{X}^{true}(\boldsymbol{\beta}) = \left.\frac{\mathrm{d}\boldsymbol{T}_2^{true}}{\mathrm{d}\boldsymbol{\beta}}\right|_{T_1} = \frac{\mathrm{d}\boldsymbol{T}_2^{true}}{\mathrm{d}\boldsymbol{W}}\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}\boldsymbol{\beta}} = \mathbf{M}(\boldsymbol{T}_1^{exact})\boldsymbol{S}_w(\boldsymbol{\beta}) \quad (23)$$

Matrix $\boldsymbol{S}_w$ of the sensitivity coefficients of transmittance W to the different components of the parameter vector is introduced:

$$\boldsymbol{S}_w(\boldsymbol{\beta}^{exact}) = \frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}\boldsymbol{\beta}}(\boldsymbol{\beta}^{exact}) = \left[\boldsymbol{S}_{w1}(\boldsymbol{\beta}^{exact}) \quad \boldsymbol{S}_{w2}(\boldsymbol{\beta}^{exact}) \quad \cdots \quad \boldsymbol{S}_{wn}(\boldsymbol{\beta}^{exact})\right] \quad (24)$$

Here, each of the column vectors $\boldsymbol{S}_{w\,j}(\boldsymbol{\beta}^{exact})$ has been presented in a concatenated form of matrix $\boldsymbol{S}_w(\boldsymbol{\beta}^{exact})$. Each of these vectors is the sensitivity coefficient of transmittance $W(t,\boldsymbol{\beta}^{exact})$ to parameter β_j at the different measurement times t_k [4]:

$$[\boldsymbol{S}_w(\boldsymbol{\beta}^{exact})]_{kj} = [\boldsymbol{S}_{wj}(\boldsymbol{\beta}^{exact})]_k = \frac{\partial W}{\partial \beta_j}(t_k; \boldsymbol{\beta}^{exact}) \text{ for } 1 \leqslant k \leqslant m \text{ and } 1 \leqslant j \leqslant n \quad (25)$$

Let us note that sensitivity matrix $\boldsymbol{X}^{exact}(\boldsymbol{\beta})$ is completely different from the sensitivity matrix of the impedance-based model (11b), which is defined by:

$$\left.\frac{\mathrm{d}\boldsymbol{T}_2}{\mathrm{d}\boldsymbol{\beta}_{z2}}\right|_{\boldsymbol{q}} = \mathbf{M}(\boldsymbol{q})\frac{\mathrm{d}\boldsymbol{Z}_2}{\mathrm{d}\boldsymbol{\beta}_{z2}} \quad (26)$$

Using the same chain derivation rule as in (23), an expression for the sensitivity matrix $\boldsymbol{X}$ present in Eq. (22b) is:

$$\boldsymbol{X}(\boldsymbol{\beta}) = \left.\frac{\mathrm{d}\boldsymbol{T}_2^{avail}}{\mathrm{d}\boldsymbol{\beta}}\right|_{\boldsymbol{T}_1^{\exp}} = \mathbf{M}(\boldsymbol{T}_1^{\exp})\boldsymbol{S}_w(\boldsymbol{\beta}) \quad (27)$$

2.4. The estimation error

One focuses here on the current value of the residual vector, whose criterion (21a) is the square of its norm, which is rewritten with a development of the vectors that correspond to both measurements:

$$\boldsymbol{r}(\boldsymbol{\beta}) \equiv \boldsymbol{T}_2^{\exp} - \boldsymbol{T}_2^{avail}(\boldsymbol{\beta}) = \boldsymbol{T}_2^{\exp} - \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}))(\boldsymbol{T}_1^{exact} + \boldsymbol{\varepsilon}_1) = \boldsymbol{T}_2^{\exp} - \boldsymbol{T}_2^{true}(\boldsymbol{\beta}) - \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}))\boldsymbol{\varepsilon}_1 \quad (28)$$

We consider now the matrix multiplier $\mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}))$ of noise $\boldsymbol{\varepsilon}_1$ in the last term of this equation. It depends on the current value of the parameter vector $\boldsymbol{\beta}$ but can be related to its exact value $\boldsymbol{\beta}^{exact}$, using the linear character of function $\mathbf{M}()$ of R^m into $\mathrm{R}^m \times \mathrm{R}^m$:

$$\mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta})) = \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}^{exact})) + \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}) - \boldsymbol{W}(\boldsymbol{\beta}^{exact})) \quad (29)$$

So, the transmittance difference in Eq. (29) can be written for a value of $\boldsymbol{\beta}$ in the neighbourhood of $\boldsymbol{\beta}^{exact}$ (first order expansion):

$$\boldsymbol{W}(\boldsymbol{\beta}) - \boldsymbol{W}(\boldsymbol{\beta}^{exact}) \approx \boldsymbol{S}_w(\boldsymbol{\beta}^{exact}))(\boldsymbol{\beta} - \boldsymbol{\beta}^{exact}) = \sum_{j=1}^{n}\boldsymbol{S}_{wj}(\boldsymbol{\beta}^{exact})\left(\beta_j - \beta_j^{exact}\right) \quad (30)$$

So, Eqs. (28)–(30) allow getting a new expression for the estimation error (22a):

$$\boldsymbol{e}_{\hat{\beta}} \equiv \hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{exact} = \mathbf{A}(\boldsymbol{\beta}^{exact})\boldsymbol{r}(\hat{\boldsymbol{\beta}}) = \mathbf{A}(\boldsymbol{\beta}^{exact})\left(\boldsymbol{T}_2^{\exp} - \boldsymbol{T}_2^{true}(\boldsymbol{\beta}^{exact}) - \boldsymbol{K}(\hat{\boldsymbol{\beta}})\boldsymbol{\varepsilon}_1\right) \quad (31)$$

with

$$\boldsymbol{K}(\boldsymbol{\beta}) = \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}^{exact})) + \mathbf{M}(\boldsymbol{S}_w(\boldsymbol{\beta}^{exact})(\boldsymbol{\beta} - \boldsymbol{\beta}^{exact})) \quad (32)$$

Eq. (31) can be also written, using (20a):

$$\boldsymbol{e}_{\beta} = \boldsymbol{A}(\boldsymbol{\varepsilon}_2 - \boldsymbol{K}(\hat{\boldsymbol{\beta}})\boldsymbol{\varepsilon}_1) \quad (33a)$$

$$\text{with } \boldsymbol{A} = \mathbf{A}(\boldsymbol{\beta}^{exact}) = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T \text{ where } \boldsymbol{X} = \mathbf{M}_{\boldsymbol{T}1}\boldsymbol{S}_w \text{ and } \mathbf{M}_{\boldsymbol{T}1} = \mathbf{M}(\boldsymbol{T}_1^{\exp}) \text{ and } \boldsymbol{S}_w = \boldsymbol{S}_w(\boldsymbol{\beta}^{exact}) \quad (33b)$$

The estimation error $\boldsymbol{e}_{\hat{\beta}} = \hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^{exact}$ is introduced in (33a), using (32):

$$\boldsymbol{e}_{\hat{\beta}} = \boldsymbol{A}(\boldsymbol{\varepsilon}_2 - \mathbf{M}_{\boldsymbol{W}}\boldsymbol{\varepsilon}_1 - \mathbf{M}\left(\boldsymbol{S}_w\boldsymbol{e}_{\hat{\beta}}\right)\boldsymbol{\varepsilon}_1) \text{ with } \mathbf{M}_{\boldsymbol{W}} = \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}^{exact})) \quad (34)$$

The last term of (34) is modified using the commutative property of the matrix form of the corresponding convolution product:

$$\boldsymbol{e}_{\hat{\beta}} = \boldsymbol{A}(\varepsilon_2 - \boldsymbol{M_W}\varepsilon_1 - \boldsymbol{M}_{\varepsilon 1}\boldsymbol{S}_w\boldsymbol{e}_{\hat{\beta}}) \quad \text{with} \quad \boldsymbol{M}_{\varepsilon 1} = \mathbf{M}(\varepsilon_1) \tag{35}$$

This allows the calculation of the estimation error:

$$\boldsymbol{e}_{\hat{\beta}} = (\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{M}_{\varepsilon 1}\boldsymbol{S}_w)^{-1}\boldsymbol{A}(\varepsilon_2 - \mathbf{M}_{\boldsymbol{W}}\varepsilon_1) \tag{36}$$

A first order series expansion, for a low norm of ε_1, of the inverse of $(\boldsymbol{I}_n + \boldsymbol{A}\boldsymbol{M}_{\varepsilon 1}\boldsymbol{S}_w)$ is implemented:

$$\boldsymbol{e}_{\hat{\beta}} \approx (\boldsymbol{I}_n - \boldsymbol{A}\boldsymbol{M}_{\varepsilon 1}\boldsymbol{S}_w)\boldsymbol{A}(\varepsilon_2 - \boldsymbol{M_W}\varepsilon_1) \approx \boldsymbol{A}(\varepsilon_2 - \boldsymbol{M_W}\varepsilon_1) \tag{37}$$

Let us note that, strictly speaking, the above series expansion should be made with dimensionless vectors, that is with the use of scaled sensitivity coefficients [4]. However, this does not modify the order of the corresponding terms.

This shows that the estimation bias is of the second order with respect to the measurement noise, even if no direct explicit expression is available:

$$\boldsymbol{b}_{\hat{\beta}} = \mathrm{E}(\boldsymbol{e}_{\hat{\beta}}) = -\boldsymbol{A}\mathrm{E}(\boldsymbol{M}_{\varepsilon 1}\boldsymbol{S}_w\boldsymbol{A}(\varepsilon_2 - \boldsymbol{M_W}\varepsilon_1)) \tag{38}$$

The variance–covariance matrix of the estimation error can be calculated:

$$\begin{aligned}\mathrm{cov}(\boldsymbol{e}_{\hat{\beta}}) = \mathrm{cov}(\hat{\boldsymbol{\beta}}) &= \mathrm{E}\left[(\boldsymbol{e}_{\hat{\beta}} - \boldsymbol{b}_{\hat{\beta}})(\boldsymbol{e}_{\hat{\beta}}^T - \boldsymbol{b}_{\hat{\beta}}^T)\right] \approx \mathrm{E}\left(\boldsymbol{e}_{\hat{\beta}}\boldsymbol{e}_{\hat{\beta}}^T\right) \\ &= \boldsymbol{A}\mathrm{cov}(\varepsilon)\boldsymbol{A}^T\end{aligned} \tag{39a}$$

with

$$\varepsilon = \varepsilon_2 - \boldsymbol{M_W}\varepsilon_1 = [-\boldsymbol{M_W} \quad \boldsymbol{I_m}]\varepsilon^* \tag{39b}$$

The expression of the variance–covariance matrix of modified noise ε can be derived:

$$\mathrm{cov}(\varepsilon) = \mathrm{E}\left[\varepsilon_2\varepsilon_2^T\right] + \boldsymbol{M_W}\mathrm{E}\left[\varepsilon_1\varepsilon_1^T\right]\boldsymbol{M_W}^T - \mathrm{E}\left[\varepsilon_2\varepsilon_1^T\right]\boldsymbol{M_W}^T - \boldsymbol{M_W}\mathrm{E}\left[\varepsilon_1\varepsilon_2^T\right] \tag{41}$$

Because of assumptions (20b) and (20c):

$$\begin{aligned}\mathrm{E}\left[\varepsilon_2\varepsilon_2^T\right] = \mathrm{cov}(\varepsilon_2) = \sigma^2\boldsymbol{I}_m; \mathrm{E}\left[\varepsilon_1\varepsilon_1^T\right] = \mathrm{cov}(\varepsilon_1) = \sigma^2\boldsymbol{I}_m; \mathrm{E}\left[\varepsilon_2\varepsilon_1^T\right] \\ = \mathrm{E}\left[\varepsilon_1\varepsilon_2^T\right] = 0\end{aligned} \tag{42}$$

Eq. (41) becomes:

$$\mathrm{cov}(\varepsilon) = \sigma^2\boldsymbol{I}_m + \boldsymbol{M_W}(\sigma^2\boldsymbol{I}_m)\boldsymbol{M_W}^T = \sigma^2\left(\boldsymbol{I}_m + \boldsymbol{M_W}\boldsymbol{M_W}^T\right) \tag{43}$$

This is substituted into (39a):

$$\mathrm{cov}(\hat{\boldsymbol{\beta}}) \approx \boldsymbol{A}\mathrm{cov}(\varepsilon)\boldsymbol{A}^T = \sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T(\boldsymbol{I}_m + \boldsymbol{M_W}\boldsymbol{M_W}^T)\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1} \tag{44}$$

which yields:

$$\mathrm{cov}(\hat{\boldsymbol{\beta}}) = \sigma^2(\boldsymbol{X}^T\boldsymbol{X})^{-1}\left[\boldsymbol{I}_n + \boldsymbol{X}^T\boldsymbol{M_W}\boldsymbol{M_W}^T\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{X}\right)^{-1}\right] \tag{45}$$

For practical reasons, one can replace, in the above expression, $\boldsymbol{\beta}^{exact}$ in the definition (34) of $\boldsymbol{M_W} \equiv \mathbf{M}(\boldsymbol{W}(\boldsymbol{\beta}^{exact}))$ by its available value, that is its estimated value, and the same is true for sensitivity matrix $\boldsymbol{X}$ of model (21b):

$$\begin{aligned}\boldsymbol{M_W} \approx \boldsymbol{M_W}^{avail} = \mathbf{M}(\boldsymbol{W}(\hat{\boldsymbol{\beta}})) \quad \text{as well as } \boldsymbol{X} \approx \boldsymbol{X}^{avail} \\ = \boldsymbol{M}_{\boldsymbol{T}1}\boldsymbol{S}_w(\hat{\boldsymbol{\beta}}) \text{ where } \boldsymbol{M}_{\boldsymbol{T}1} = \mathbf{M}(\boldsymbol{T}_1^{\exp}))\end{aligned} \tag{46}$$

3. Application of the transmittance-based parameter estimation technique

3.1. The three-layer method and the associated decrease in the number of its unknowns

As an example, the preceding transmission-based estimation technique will be applied to the measurement of thermal diffusivity of light insulating materials that has already been developed in our laboratory, the three-layer method [9].

Fig. 4, already displayed above, gives the scheme of the experimental setup: the sample to be characterized, of low thickness e, of conductivity λ and volumetric heat capacity ρc, is set in between two brass slabs and a planar heating resistor, located below the lower slab, allows creating a surface heat source $q(t)$. This source is converted into a flux that enters the lower brass slab, with a flux in the opposite direction whose value is limited by the presence of an insulating material. Temperatures of the lower and upper brass samples, respectively T_1 for the front face and T_2 for the rear face, are measured by two thermocouples and recorded.

We assume here that heat transfer is 1D, that is that the heat transfer coefficient h_3 characterizing the lateral convection and linearized radiation losses does not affect the response of the thermocouple hot junctions located close to the vertical symmetry axis of the system and that both the heating source and the pressure on the assembly are uniforms in the corresponding contact plane.

With this 1D assumption, it is possible to demonstrate the advantage of working with transmittance-based models rather than with impedance-based ones.

The quadrupole modelling [12] is used here. Since the thermal resistances of the brass slabs, numbered 1 (front face) and 2 (rear face), of thicknesses $e_{brass\,i}$ and volumetric capacity ρc_{brass}, can be neglected, these are modelled by thermal capacities $C_i = \rho c_{brass} e_{brass\,i}$, for i = 1 or 2, while heat losses on the corresponding external faces are modelled by heat transfer coefficients h_1 and h_2. Contrary to what is shown in Fig. 4, we assume here, for simplicity reasons that will not affect our conclusion, that no insulating material for limiting the losses over the lower (front) brass sample is present in the setup stack: this allows the use of a convecto-radiative transfer coefficient h_1 with the surrounding environment.

In the Laplace domain, calling p the Laplace parameter, the solution of this problem is given by:

$$\begin{bmatrix}\overline{T}_1 \\ \bar{q}\end{bmatrix} = \begin{bmatrix}1 & 0 \\ h_1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ pC_1 & 1\end{bmatrix}\begin{bmatrix}A & B \\ C & D\end{bmatrix}\begin{bmatrix}1 & 0 \\ pC_2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 \\ h_2 & 1\end{bmatrix}\begin{bmatrix}\overline{T}_2 \\ 0\end{bmatrix} \tag{47a}$$

or, after a development:

$$\begin{bmatrix}\overline{T}_1 \\ \bar{q}\end{bmatrix} = \begin{bmatrix}1 & 0 \\ h_1 + pC_1 & 1\end{bmatrix}\begin{bmatrix}A & B \\ C & D\end{bmatrix}\begin{bmatrix}1 & 0 \\ h_2 + & pC_2\end{bmatrix} \tag{47b}$$

$$\begin{bmatrix}\overline{T}_2 \\ 0\end{bmatrix} = \begin{bmatrix}A + BH_2 & B \\ C + AH_1 + DH_2 + BH_1H_2 & D + BH_1\end{bmatrix}\begin{bmatrix}\overline{T}_2 \\ 0\end{bmatrix} \tag{47c}$$

where:

$$\begin{aligned}&H_1 = h_1 + pC_1; \quad H_2 = h_2 + pC_2; \quad a = \frac{\lambda}{\rho c} \\ &A = \cosh\left(e\sqrt{p/a}\right); \quad B = \sinh\left(e\sqrt{p/a}\right)/(\lambda\sqrt{p/a}) \\ &C = \lambda\sqrt{p/a}\sinh\left(e\sqrt{p/a}\right); \quad D = A\end{aligned} \tag{47d}$$

Solution of the direct problem, in terms of front and rear face temperature responses to $q(t)$ results from numerical inversion [13] of the Laplace transforms of both surface temperatures:

$$\overline{T}_1 = \frac{A + BH_2}{C + A(H_1 + H_2) + BH_1H_2}\bar{q} = \overline{Z}_1\bar{q} \tag{48}$$

$$\overline{T}_2 = \frac{1}{C + A(H_1 + H_2) + BH_1H_2}\bar{q} = \overline{Z}_2\bar{q} \tag{49}$$

Of course, the operational impedances $\overline{Z}_1$ and $\overline{Z}_2$, that is the factor of $\bar{q}$ in the above two expressions, of the same form as Eqs. (9a) and (9b), can be recognized. The ratio of the two expressions makes the operational transmittance $\overline{W}_{21}$, noted $\overline{W}$ here, appear:

Table 1
Values of the different parameters of the impedance-based model for the rear and front face temperatures T_2 and T_1.

Quantity	Sample conductivity	Sample volumic heat	Sample thickness	Heat loss coefficients	Heat capacities of brass layers by unit area	Surface heat source strength	Duration of the excitation
Symbol	λ	ρc	e	$h_1 = h_2$	$C_1 = C_2$	q_0	t_0
Unit	$W\,m^{-1}\,K^{-1}$	$J\,m^{-3}\,K^{-1}$	m	$W\,m^{-2}\,K^{-1}$	$J\,m^{-2}\,K^{-1}$	$W\,m^{-2}$	s
Value	0.025	$4.0\ 10^4$	$6.41\ 10^{-3}$	5	1292	2840.4	10

$$\overline{T}_2 = \frac{1}{A + BH_2}\bar{q} = \overline{W}\ \overline{T}_1 \tag{50}$$

If one refers to Eq. (10), the operational transmittance $\overline{W}$ depends on three parameters only, two of them being dimensionless and the third one having the dimension of a time. These can be gathered in a parameter vector $\boldsymbol{\beta}$ of a model of the B type using a transmittance transfer function:

$$T_2(t;\boldsymbol{\beta}) = W(t;\boldsymbol{\beta}) * T_1(t) \quad \text{where } \boldsymbol{\beta} = \left[\frac{a}{e^2}, \frac{h_2 e}{\lambda}, \frac{C_2}{\rho c e}\right]^T \tag{51a, b}$$

Let us note that $\boldsymbol{\beta}$ can be defined by any other independent combination of the three same parameters.

A close look at the expressions of $\overline{Z}_2$ shows that it depends on parameter vector $\boldsymbol{\beta}_{z2}$ which includes three more parameters, the front face Biot number h_1e/λ, a capacity ratio $C_1/(\rho ce)$ linked to the brass capacity C_1 of the front face and a parameter having a physical dimension, the thermal resistance e/λ of the sample:

$$\boldsymbol{\beta}_{z2} = \left[\frac{a}{e^2}, \frac{h_2 e}{\lambda}, \frac{C_2}{\rho c e}, \frac{h_1 e}{\lambda}, \frac{C_1}{\rho c e}, \frac{e}{\lambda}\right]^T \tag{51c}$$

or any combination of the same parameters.

If the intensity of the source is unknown, we are in the A3 type of problem, and the transient electrical power has to be measured in order to get the $\boldsymbol{\beta}_{z2}$ parameters detailed in Eq. (51c).

In a A2 type configuration, the time shape of the source intensity is known, within a proportionnality factor q_0, that is $q(t) = q_0 f(t)$, where $f(t)$ is for example a door function, see Eq. (53) further down, and the corresponding model becomes:

$$T_2(t, \boldsymbol{\beta}_2) = Z_2(t, \boldsymbol{\beta}_{z2})^* q_0 f(t) \tag{52a}$$

where:

$$\boldsymbol{\beta}_2 = \left[\frac{a}{e^2}, \frac{h_2 e}{\lambda}, \frac{C_2}{\rho c e}, \frac{h_1 e}{\lambda}, \frac{C_1 e}{\lambda}, \frac{q_0 e}{\lambda}\right]^T \tag{52b}$$

or any combination of the same parameters. Let us note that the number of parameters in $\boldsymbol{\beta}_2$, Eq. (52b), is the same as in $\boldsymbol{\beta}_{z2}$, Eq. (51c), because its 6th parameter group e/λ has been replaced by q_0e/λ.

So, when two temperatures are measured, that is in the B case, the corresponding model (51a) is more parsimonious than the classical impedance models of the A1 type or of the A2 type (three parameters less).

In fact, in this example, whatever the model used, three parameters can be supposed to be known without any need for their estimation: C_1, C_2 and e. So the transmittance method of the B type requires estimation of 3 parameters (a, λ, h_2) instead of four parameters (a, λ, h_2, h_1) for the impedance model of the A1 type and five parameters $(a, \lambda, h_2, h_1, q_0)$ for the A2 type. This is made through the minimization of the least square sum (19) for $\boldsymbol{\beta} = [a \quad \lambda \quad h_2]^T$: this is either a 25% or a 40% reduction in the number of parameters using a single experimental temperature output response, which tends to reduce the standard deviation of the estimate in a substantial way, see [6] and the comparative study made in Section 3.3 further down.

3.2. Monte Carlo versus stochastic parameter estimation error in the transmittance model for the three-layer method

We have selected the parameter values shown in Table 1 to model the exact output of the T_2 signal for a door excitation of duration t_0:

$$q(t) = q_0[\mathrm{H}(t) - \mathrm{H}(t - t_0)] \tag{53}$$

where H () is the Heaviside function.

The T_1 and T_2 temperature signals have been sampled for m equally spaced times over a duration $t_m = t_{\max}$.

A noised synthetic signal has been generated by addition of a normal independent identically distributed noise of zero mean and standard deviation σ according to Eqs. (20a) and (20b). This procedure has been repeated N times in order to simulate N experiments in a Monte Carlo type process. For each experiment (numbered i), iterative minimization of least square sum (19) by the Levenberg–Marquardt algorithm on the $[t_1 \quad t_m]$ interval has generated an estimation $\hat{\beta}_j^{(i)}$ for each of the three parameters (j = 1 to 3). The characteristics of these estimations are given in Table 2.

An example of such synthetic "experimental" temperature responses is given in Fig. 5. The temperature residuals $\boldsymbol{r}(\boldsymbol{\beta}) = \boldsymbol{T}_2^{\exp} - \mathbf{M}(\boldsymbol{W}(\hat{\boldsymbol{\beta}}))\boldsymbol{T}_1^{\exp}$ are also plotted in the same figure.

Table 3 shows the results of these inversions, in terms of averaged $\bar{\hat{\beta}}_j$ and exact β_j^{exact} (same as in Table 1) values of each parameter and in terms of their statistical $s_{\hat{\beta}_j}$ and stochastic standard deviations:

$$\bar{\hat{\beta}}_j = \frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_j^{(i)}; \quad s_{\hat{\beta}_j}^2 = \frac{1}{N}\sum_{i=1}^{N}\hat{\beta}_j^{(i)^2} - \bar{\hat{\beta}}_j^2; \quad \sigma_{\hat{\beta}_j}^2 = [\mathrm{cov}(\hat{\boldsymbol{\beta}})]_{jj} \tag{54}$$

where the variance–covariance matrix $\mathrm{cov}(\hat{\boldsymbol{\beta}})$ is calculated using Eqs. (45) and (46).

These parameters have been deduced from the three parameter groups of parameter vector $\boldsymbol{\beta}$ with the assumption that the thickness e of the sample and the thermal capacity C_2 of the second brass layer are both known.

The last column in Table 3 corresponds to the estimated volumtric capacity $\rho c = \lambda/a$ of the sample: it is just deduced from the ratio of the a and λ estimations, for each simulation. Selection of a different combination of the three parameters to be estimated (the coefficients of the parameter vector $\boldsymbol{\beta}$) out of the four parameters a, λ, h_2 and ρc, with a later deduction of the remaining 4th one does not change their values, see [6].

Several remarks can be made here about these results:

- the estimation bias, evaluated through the N Monte Carlo simulations of inversion by the difference between lines 2 and 1 in this Table is nearly equal to zero: if this bias is normalized by the exact value of each of the three parameters, one can see that its relative value is less than 0.034%;
- even if it is not perfect, the Monte Carlo (statistical) standard deviation of these parameters is very close to its stochastic counterpart: this shows that our variance–covariance matrix derivation (45) is pertinent and can allow to bypass the lengthy Monte Carlo simulation process to assess the precision of the estimates in a real transmittance-based estimation;

Table 2
Characteristics of the N synthetic measurements.

Parameter	Number of times of measurement	Number of synthetic measurements	Duration of the estimation interval	Standard deviation of the noise of T_2 and T_1
Symbol	m	N	$t_m = t_{\max}$	σ
Value	1000	10,000	200 s	0.01 K

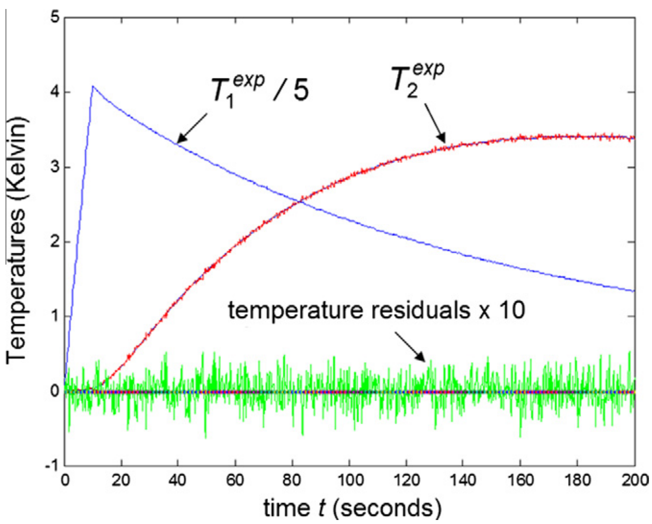


Fig. 5. Three-layer method: synthetic temperature responses and least squares residuals.

-t-

he relative standard deviation of the estimated parameters, either statistical or stochastic, is lower than 1%, for a signal over noise ratio $T_2^{\exp\max}/\sigma$ for the $T_2^{\exp}$ temperature response of the order of 330: this means that precise measurements of the two temperatures, in a 3-layer method experiment for characterizing a light insulating sample, does not lead to a too high amplification of the measurement noise. This allows a precise characterization of the two thermal parameters of the sample. Of course, this conclusion relies on the assumption that the original impedance model (48) and (49) used for deriving the expression of the impedances is a good representation of the real experiment;

3.3. Relative perfomances of the parameter estimations by the transmittance model and by the impedance models for the three-layer method

Comparison between the results of the estimation techniques based on the models of the B type (transmittance), and of the A1 and A2 types (impedance) are presented in Table 3 in terms of relative standard deviation of the estimated parameters calculated on stochastic bases, that is through Eqs. (45) or (13).

- line 1 in Table 4 is just a duplication of line 3 in Table 3 and relates to the estimation of 3 parameters by the B type model.
- line 2 in Table 4 corresponds to the same configuration, but the stochastic standard deviations of the three estimated parameters is calculated by Eq. (45) where the second term in the right member of this equation is omitted: this corresponds to a case where noise only affects measured temperature $\boldsymbol{T}_2^{\exp}$ while the other temperature is measured without any noise ($\boldsymbol{T}_1^{\exp} = \boldsymbol{T}_1^{exact}$). Comparison of lines 1 and 2 shows that presence of noise in the measurement of this pseudo-causal signal implies only a 0.5% increase of the standard deviation of the four parameters that are looked for at most (see columns 3 to 5, and 8). In other configurations, such as a sample made out of a heavy insulating material (PVC type), the corresponding increase of these standard deviations can reach 3%. The same effect appears when the original thickness (e_{brass} = 0.4 mm) of the two metallic slabs is changed into 4 mm: the corresponding increase of the standard deviations becomes equal to 9%. A similar increase is observed if the duration of the estimation interval is increased. This moderate contribution of the second term of (45), which does not exceed 10% (in terms of standard deviation increase) under realistic conditions, stems from the fact that the experimental excitation $T_1^{\exp}$ is convoluted with a transmittance function that damps its noisy component.
- inversion of the single $\boldsymbol{T}_2^{\exp}$signal by transmittance model (49) of the A1 type, with 4 parameters is not possible since the standard deviations of the estimates of h_1 and h_2 become very large: this is due to the ill-conditioning of the information matrix $\boldsymbol{S}^T\boldsymbol{S}$ (once the sensitivity coefficients which forms the columns of the sensitivity matrix $\boldsymbol{S}$ have been scaled by the exact values of the corresponding parameters). This is caused by the fact that the sensitivity coefficients to h_1 and h_2 are nearly proportional. Consequently the model has only 3 and not 4 degrees of freedom. That is why assumption $h_1 = h_2$ is perfectly legitimate. The relative standard dispersions calculated for the three remaining parameters with this assumption are shown in line 3 of the table: comparison with results of the estimation of B type, also with 3 parameters (line 1) shows that there is a 8% reduction of the relative standard deviation of the conductivity (from a value of 0.08%) and a 6% increase of the relative dispersion of the diffusivity (from a value of 0.26%).

So the transmittance method (B type) is more appropriate for estimating the diffusivity here (and also the volumetric heat, see column 8) than the impedance method with perfectly know excitation (A1 type) while the opposite is true in terms of conductivity estimation.

Table 3
Comparison between Monte Carlo estimation and exact or stochastic characteristics of the estimates for the transmittance method (B type model).

	Number j of parameter Parameter Unit		1 $\beta_1 = a$ 10^{-7}m^2. s^{-1}	2 $\beta_2 = \lambda$ W m^{-1} K^{-1}	3 $\beta_3 = h_2$ W m^{-2} K^{-1}	$\rho c = \lambda/a$ J m^{-3} K^{-1}
1	Exact or averaged value	Exact β_j^{exact}	6.2500	0.0250	5.0000	40,000
2		Statistical $\bar{\hat{\beta}}_j$	6.2521	0.0250	5.0017	40,027
3	Scaled standard deviations (%)	Stochastic $\sigma_{\hat{\beta}_j}/\beta_j^{exact}$	0.2588	0.0885	0.2326	0.3400
4		Statistical $s_{\hat{\beta}_j}/\beta_j^{exact}$	0.2588	0.0879	0.2311	0.3390

Table 4
Comparison of the performances of the estimation methods of the B, A1 and A2 in terms of relative stochastic standard deviation $\sigma_{\hat{\beta}_j}/\beta_j^{exact}$ of the estimated parameters (%) for a temperature noise of standard deviation $\sigma = 0.01$ K for $\boldsymbol{T}_2^{\exp}$.

	Type of model and equation for the standard deviation	Number of parameters to be estimated	$\beta_1 = a$	$\beta_2 = \lambda$	$\beta_3 = h_2$	$\beta_4 = h_1$	$\beta_5 = q_0$	$\rho c = \lambda/a$
1	B with Eq. (45) same noise for $\boldsymbol{T}_2^{\exp}$ and $\boldsymbol{T}_1^{\exp}$	3	0.2588	0.0885	0.2326			0.3400
2	B with a zero noise for $\boldsymbol{T}_1^{\exp}$	3	0.2584	0.0882	0.2319			0.3395
3	A1 with Eq. (13) with assumption $h_1 = h_2$	3	0.274	0.0813	0.107			0.347
4	A2 with Eq. (13) with assumption $h_1 = h_2$	4	0.539	3.553	1.805		3.245	4.031
5	A2 with Eq. (13)	5	0.539	3,553	$3 \cdot 10^6$	$3.\ 10^6$	3.245	4.031

The results impedance method of the A2 type (known shape of the intensity of the heat source but unknown proportionality coefficient q_0) are also shown in the same table. With the assumption $h_1 = h_2$ (line 4), the 4 parameters can be estimated, but the relative standard deviations of the estimated parameters display dispersions that are twice as high for the diffusivity, when compared with the B type of estimation (line 1). These estimations are even 40 times higher for thermal conductivity and 12 times higher for the reconstructed volumetric heat. Of course, no assumption about the two heat losses coefficients leads to an impossibility of recovering all the values of the five parameters (see line 5) for the same reason as for the A1 type method. However estimations of the three other coefficients, a, λ and q_0 are the same, because sensitivity to the heat loss coefficients are low, since the value of $h_1 = h_2$ is low with respect to the heat capacities $C_1 = C_2$ of the brass layers in Eqs. (47c) and (49).

In practice, the A1 assumption, that is a perfectly known $q(t)$), is unrealistic. In the absence of the possibility of direct or indirect measurement of this excitation, the A2 estimation method (known time shape of $q(t)$, with an unknown q_0 factor) seems to constitute a reasonable compromise if an impedance type estimation technique has to be implemented: here the operator switches the 'on' (at time $t = 0$) and 'off' (at time $t = t_0$) knob of the resistor: a perfectly known door function surface heat source (54) can be assumed if:

– the heat capacity of the heating probe is neglected,
– the variation of its electric resistance with temperature is neglected,
– the heat losses to the upstream part of the 1D thermal circuit are neglected: they concern either the insulating material, or the h_2 coefficient in the simplified modelling made in Section 3.1 and shown in Fig. 4. This requires the knowledge of four extra parameters: the thickness of this material, as well as its conductivity, its volumetric heat and the boundary condition of the opposite face.

4. Transmittance-based models and their parsimonious character

One of the most effective reason for an impossible estimation of the parameters of a model lies in its lack of parsimony. So a general comparative study of the numbers of parameters of the transmittance method and of the more classical impedance method will be made here.

We restrict this study to the problem of characterizing a sample made out of a homogeneous material using a uniform surface transient heat source $q(t)$(W) with one or two possible measurements, either the rear face temperature $T_2(t)$ (temperature on the face opposite to the source, impedance method) or both the rear face temperature as well as the front face temperature $T_1(t)$ (transmittance method), see Fig. 6a. The temperature of the outside environment is assumed to be equal to the initial temperature of the whole sample/front and rear face material system and is taken as a reference zero temperature.

This analysis is valid for a corresponding heat transfer model in any one-dimensional case, that is either for a one-directional planar sample (slab) or for a cylindrical or spherical sample.

The sample transfer matrix present in the thermal network in Fig. 6b is composed of the four coefficients A, B, C and D already given in Eq. (47c) for a planar geometry. Apart from the Laplace parameters, these coefficients depend on two parameter groups, the sample characteristic frequency a/e^2 and the sample thermal resistance e/λ (for a unit area) or any other combination of these groups. In this figure this sample quadrupolar matrix [12] is surrounded by a network corresponding to:

– the front face system (for example, a brass layer and conducto-radiative losses represented by a h_1 coefficient in the three-layer model depicted in Section 3.1) and the stimulation system where the $q(t)$ power is dissipated. This heating probe may have its own thermal inertia and thermal resistance;
– the rear face system (for example, a brass layer and conducto-radiative losses represented by a h_1 coefficient in the previous three-layer model).

Both systems can be represented by a 2×2 matrix: this yields a global quadrupolar equation formed by the front face/sample/rear face setup involving three matrices such as Eq. (47b).

The A1 estimation technique is first considered: calculation of the rear face temperature response $T_2(t)$, starting from a perfectly known heat source $q(t)$, requires the knowledge of the two independent parameters of the sample, as well as the knowledge of n_F and n_R independent parameter groups present in the coefficients of the front face and of the rear face matrix respectively. So the to-

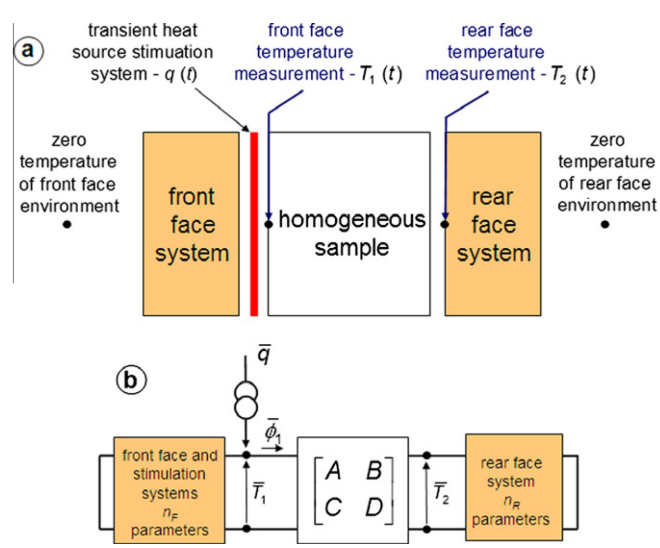


Fig. 6. Counting of the number of parameters of impedance-based and transmittance-based characterization methods: (a) homogeneous sample, and its stimulation and measurement systems; (b) simplified thermal network of the possible measurement configurations.

tal number of parameters of the heat transfer model used by this technique is:

$$n_{A1} = n_F + n_R + 2 \tag{55}$$

In the A2 estimation technique, the same n_{A1} independent parameter groups associated to impedance Z_2 are present, with an additional multiplicative constant q_0 present in the definition of $q(t)$: $q(t) = q_0 f(t)$. However, this constant merges with one of the n_F parameter groups present also as a multiplicative constant and, finally, the number of independent parameter groups of the model remains unchanged:

$$n_{A2} = n_{A1} = n_F + n_R + 2 \tag{56}$$

Let us note that a physical explanation can be given for this absence of increase in the number of parameter groups: impedance Z_2 is expressed in K W^{-1} and has the units of a thermal resistance. As a consequence one of its parameter groups, a multiplicative constant (in the Laplace domain) has the same unit and merges with q_0(W) to get a product T_2 in kelvins since $f(t)$ is dimensionless.

We do not deal with the number of parameters of the models of the A3 type, where $q(t)$ is completely unknown, because it requires a separate measurement of this input, which is not within the scope of this study.

In the B estimation technique, where both experimental temperatures T_1 and T_2 linked by a transmittance transfer function W are measured, it can be shown that this transmittance function does not depend on the parameters of the front face system. It depends only on the 2 independent parameter groups of the sample as well as on the n_R parameter groups in the rear face matrix. However, since W should be dimensionless, there is also a merge of one independent non dimensionless parameter group of the sample with a corresponding group of the rear face matrix and, consequently, a decrease by one unit of the number of independent parameters of the B model:

$$n_B = n_R + 1 \tag{57}$$

A last interesting case, called A' here, can also be considered: it corresponds to the simultaneous measurement of the rate of heat flow $\phi_1(t)$, see Fig. 6b, entering the sample (W), by a transient fluxmeter, together with the rear face temperature response T_2. The corresponding model can be written in the Laplace domain:

$$\overline{T}_2 = \overline{Z}'_2 \bar{\phi}_1 \tag{58a}$$

It can be shown that the number of independent groups of parameters present in impedance Z'_2 and consequently in this model is:

$$n_{A'} = n_R + 2 \tag{58b}$$

Let us note that there is one more parameter present in the A′ model than in the B model: this extra parameter stems from the fact that impedance Z'_2 has to be used to transform an input ϕ_1 in watts into an output T_2 in kelvins while this is not the case for the dimensionless transmittance W.

So T_1 to T_2 models using transmittance transfer functions (B type) are always more parsimonious than q to T models using impedance transfer functions (A1 or A2 type), with a difference in the number of independent parameter groups equal to $n_F + 1$. This conclusion is also valid if the transmittance-based models are compared with impedance-based model where the input is not the source q anymore but the rate of heat flow ϕ_1 entering the sample (A' type), where this difference becomes equal to unity.

We have made no assumption in the above comparison between the B, A and A' techniques about the possible intrusive character of the heating probe since it can be taken into account in the front face quadrupolar representation. However temperature sensors in all these techniques and the flux sensor in the A' technique have been supposed to be non intrusive. If this is not the case, the thermal parameters of these sensors have to be taken into account in the study of the parsimony of the corresponding models.

5. Conclusions

It has been shown in Section 2.1 that methods consisting in measuring a temperature T in two separate locations generate a model that relies these two temperatures through a dimensionless transfer function, a *transmittance*, in a convolution product. These methods can apply to different linear thermal systems with structural coefficients that do not depend on time. Both measurement points have to be selected in order to respect some conditions (causality problem): a temperature response "close" to the source (a reference response) can replace it for explaining a response at a more "'distant" point through another convolution product based on a dimensionless transfer function of another kind, called a transmittance here.

When compared to more classical models of the impedance type, where temperature at a single point is measured as a consequence of a transient heat source that is supposed to be known, possibly within a multiplicative constant, we have shown, in Section 4 for 1D configurations, that these transmittance models display a significant advantage: by construction, the number of parameters present in the transmittance function of the "distant" point response is lower than the number of parameters present in the corresponding impedance model. This type of model is more *parsimonious* than any model of the impedance type.

Of course this advantage can be used only if both temperatures can be measured without any bias and without any intrusive character that would modify heat transfer in the sample. These conditions are difficult to meet in the flash method (with a radiative excitation) for measuring a pseudo-causal front face temperature (a thermocouple is intrusive and a radiative measurement requires the knowledge of the front face emissivity in the detector spectral interval over relatively large temperature excursions.

The major difference with impedance type methods is that the new temperature "pseudo source", which allows the calculation of the distant point response, is measured with the presence of a corresponding noise. A close analysis of the least-square estimation procedure has allowed here to retrieve here the stochastic properties of the estimations of the parameters present in the transmittance function.

Results of such an estimation procedure have been applied to the temperature responses of the three-layer method for thermal characterization of light insulating materials [9]. The solutions of the direct problem are analytical in the Laplace domain here. Monte Carlo simulations of the experiment have shown that the preceding stochastic analysis was pertinent and that the noise present in the reference response did not produced any significant bias.

The noise of the pseudo-causal temperature signal affects the estimated parameters in a very weak way, which accounts for less than 10% of their value in this configuration.

In the same example, estimations using either the transmittance model with two temperature measurements or the impedance model with a single temperature measurement assuming a perfectly known intensity of the source, compete equally, in terms of dispersion around the true mean value of the parameters. Both models are adapted for providing a different thermophysical parameter with a high enough precision.

However, in the same configuration, the transmittance model outperforms the impedance model with a single temperature measurement and where the intensity of the source is known within a multiplicative constant, in terms of dispersion of the estimates around their exact values.

Without any real measurement of the power source, it is difficult to check the validity of its time shape in the impedance type estimation technique where a single temperature output is measured. This may lead to estimation biases. By contrast, this constitutes an advantage of the transmittance technique: any kind of source variation with time can be used and it is not necessary to know it since it is replaced by the pseudo-input, the temperature close to the source, which is measured and not biased consequently. Of course the errors of the estimated parameters present in parameter vector $\boldsymbol{\beta}$ still depend on the input.

In order to get the same quality in the estimation error for the impedance type techniques as in the transmittance-based method, it is necessary to measure not only the temperature input, but also the heat source, which is not always an easy task (methods of the A3 type).

Transmittance models can be extended to similar linear transfer problems with time independent coefficients where the potential is not temperature anymore, such as advection–diffusion problems in mass transfer for a source reconstruction [14] where concentration potentials and mass flow rates are used. Other applications based on the measurement of two potentials can be quoted, such as the characterization of the transient flow of a liquid through a porous sample, see the *step decay* method [15] which corresponds to a non linear model.

References

[1] A. Degiovanni, A. Laurent, Une nouvelle technique d'identification de la diffusivité thermique pour la méthode flash (A new identification technique of thermal diffusivity in the heat pulse method, in French), Revue de Physique Appliquée 21 (1986) 229–237.

[2] X. Zhang, A. Degiovanni, Mesure de l'effusivité thermique de matériaux solides et homogènes par une méthode de ''sonde'' plane (Measurement of thermal effusivity of solid materials through a planar probe, in French), Journal de Physique III 6 (1993) 1243–1265.

[3] T. Boumaza, J. Redgrove, Use of the transient plane source technique for rapid multiple thermal property measurements, International Journal of Thermophysics 24 (2) (2003) 501–512, http://dx.doi.org/10.1023/A:1022928206859.

[4] J.V. Beck, K.J. Arnold, Parameter estimation in engineering and science, Wiley, New York, 1977.

[5] O. Fudym, J.-C. Batsale, Introduction to linear least squares estimation and ill-posed problems for experimental data processing, in: H.R.B. Orlande, O. Fudym, D. Maillet, R.M. Cotta (Eds.), Thermal Measurements and Inverse Techniques, CRC Press, Taylor & Francis Group, Bocca Raton, 2011, pp. 233–281.

[6] B. Rémy, S. André, S., D. Maillet, Non linear parameter estimation problems: tools for enhancing metrological objectives, in: Proceeding of Eurotherm Seminar 94: Metti 5 Advanced Spring School ''Thermal Measurements and Inverse Techniques, Roscoff, Brittany, France, June 13-18, 2011, free download at http://www.sft.asso.fr/document.php?pagendx=12299&project=sft.

[7] A. Kavianapour, J.V. Beck, Thermal property estimation utilizing the Laplace transform with application to asphaltic pavement, International Journal of Heat and Mass Transfer 20 (1977) 259–267.

[8] D. Hadisaroyo, D., J.C. Batsale, J.-C., Degiovanni, A., Un appareillage simple pour la mesure de la diffusivité thermique de plaques minces (A simple apparatus for the measurement of the thermal diffusivity of thin films, in French),), Journal de Physique III France 2, (1), (January 1992), 111-128, DOI http://dx.doi.org/10.1051/jp3:1992259.

[9] Y. Jannot, A. Degiovanni, G. Payet, Thermal conductivity measurement of insulating materials with a three layers device, International Journal of Heat and Mass Transfer 52 (2009) 1105–1111.

[10] W.N. dos Santos, Gregorio Rinaldo Jr., Hot-Wire Parallel Technique: A new method for simultaneous determination of thermal properties of polymers, Journal of Applied Polymer Science 85 (2002) 1779–1786.

[11] A. Fguiri, N. Daouas, N. Borjini, M.-S. Radhouani, H. Ben Äissia, Experimental inverse analysis for the determination of boundary conditions in the parallel hot wire technique, Experimental Thermal and Fluid Science 31 (2007) 209–220.

[12] D. Maillet, S. André, J.-C. Batsale, A. Degiovanni, C. Moyne, Thermal Quadrupoles – Solving the Heat Equation through Integral transforms, Wiley, Chichester, 2000. ISBN 0 471 98320 9.

[13] F.R. de Hoog, J.H. Knight, J.A.N. Stokes, An improved method for numerical inversion of Laplace transforms, SIAM Journal of Scientific Computing 3 (3) (1982) 357–366.

[14] T. Maalej, D. Maillet, J.-R. Fontaine, Estimation of position and intensity of a pollutant source in channel flow using transmittance functions, Inverse Problems 28 (2012) 14, http://dx.doi.org/10.1088/0266-5611/28/5/055010. 055010.

[15] D. Lasseux, Y. Jannot, Measurement of Parameters Linked to the Flow of Fluids in a Porous Material, Patent WO/2011/089367, 2011.